Kinetic and Mean-Field Game Models of Information Propagation

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INTRODUCTION:

- Vast body of literature dedicated to study of information flow through networks
- "Information" ∈ \{wealth, opinions, spins, disease, ... \}
- Recently popularized by D. Aldous as "Finite-Markov Information-Exchange (FMIE) Processes"
Components of Model

\[ \rightarrow \text{Agents } i \in G \text{ (vertices)} \]

\[ \rightarrow \text{Strength of relationship } \lambda_{ij} \geq 0 \text{ (edges)} \]

\[ \rightarrow \text{Information state } \theta_i \in \Theta, \text{ which we call "type of agent } i". \]

\[ \text{(Value at vertex } i) \]

Dynamics governed by random matching of agents with rates \( \lambda_{ij} \).

(Can be generalized to N-ary matchings.)
SCHEMATIC:

Network Geometry \( \{ \lambda_{ij} \}, i, j \in G \)

Interaction laws / Update rules (includes control policies)

Initial conditions for dist. of types in population, forcing, etc.

- We consider \( |G| \to \infty \), \( \lambda_{ij} = \lambda \) const. \( \forall i, j \in G \)

\( \implies \) Mean-field
Let $\mu(t, d\theta) = \{ i \in G : \Theta_i(t) \leq d\theta \}$ where $|G| = 1$.

**Dist. of Types in Population** $\Rightarrow$ **Law of Any Individual Agent's Type**

**Kac Master Eqn.** (N-Any Interactions)

\[
\frac{\partial}{\partial t} \int \mu(d\theta_1) \phi(\theta_1) + 2\lambda \int \Phi_{\mu}(d\theta_1) \phi(\theta_1) = \lambda \int \int_{\mathbb{R}^N} \mu(d\theta_1) \cdots \mu(d\theta_N) \mathbb{E} \left[ \Theta_i'(\theta_1, \ldots, \theta_N) \right] \]

**Type of $i$ After Interaction**

**Ave. w.r.t. Randomness in Interaction**
Reduces to Maxwell-type model for particular instances of interaction laws, like
\[ \Theta_i' = \sum_{j=1}^{N} A_{ij} \Theta_j', \quad i \in \{1, \ldots, N\}. \]

**Ex.** (Information Aggregation)

\[ A_{ij} \equiv 1 \implies \Theta_i' = \sum_{j=1}^{N} \Theta_j'. \]


  \[ \Rightarrow \text{Pricing of financial assets in OTC markets.} \]

  \[ \Rightarrow \text{Bayesian interpretation: perfect learning.} \]

  **Non-trivial dist. for** \( A_{ij} \) **\Rightarrow imperfect/biased learning.**
Many other examples have been considered:

Consensus-seeking:
\[ A_{ij} = \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}, \quad \lambda \in (0,1) \]

Randomized public goods games:
\[ A_{ij} = \begin{pmatrix} (1-\frac{1}{2}\lambda)k & \frac{1}{2}\lambda k \\ \frac{1}{2}\lambda k & (1-\frac{1}{2}\lambda)k \end{pmatrix}, \quad \lambda \in [0,1], \quad k \in [0,\infty) \]

- Boyko V - Cercignani - Gamba (2009), Boyko V - Winkler (2011)

Wealth distribution, opinion dynamics:

- Toscani, Mateus, Duering, Pareschi, and many others...
Asymptotic behavior $\rightarrow$ Stationary states
$\rightarrow$ Self-similarity, under dynamic scaling
$\rightarrow$ Convergence rates

In particular, interplay between interactions and initial data leads to interesting self-similar states:

$$\partial_t \mu + \lambda \mu = E[\mu(A_1, \cdot) \ast \cdots \ast \mu(A_N, \cdot)](\cdot)$$

$\Rightarrow$ Under rescaling, $\mu$ converges to a scaled mixture of stable laws (heavy-tailed)

(can be thought of as the dist. of a random sum of r.v.'s.)

Applications of theory to particular models of interest

→ Work in progress.

For now, will discuss a particular instance
of a kinetic model which appears naturally
in a mean-field game...
PART II:

A MEAN-FIELD GAME MODEL
OF INFORMATION PROPAGATION...
Background:

- Agents ≠ Particles

- Each agent chooses a (selfish) strategy to maximize/minimize their individual utility/cost, given present state of others.

  → Optimal control or stopping.

- Solutions are Nash equilibria (typically not unique!)
**Model Setup:**

Recall Information Aggregation (Binary)

\[
\begin{align*}
\text{Prior Types} & : \\
\theta_1 & \rightarrow \theta_1' = \theta_1 + \theta_2 \\
\theta_2 & \rightarrow \theta_2' = \theta_1 + \theta_2
\end{align*}
\]

**Bayesian Framework:**

- **Duffie - Giroux - Manso** (2010)

Consider a good with value \( X \in \{ H, L \} \)

Assume \( P(X = H) = P(X = L) = \frac{1}{2} \).
• $X$ unknown.

• Signals $\{s_k\}_{k=1}^\infty$: conditionally i.i.d. given $X$

• $s_k$ correlated to $X$ (informative)

\[
P(s_k \in ds \mid X = H)
\]

\[
P(s_k \in ds \mid X = L)
\]
Log-likelihood ratio given \( \{s_1, \ldots, s_m\} \):

\[
\begin{align*}
\log \frac{P(X=H | s_1, \ldots, s_m)}{P(X=L | s_1, \ldots, s_m)} &= \log \frac{P(X=H)}{P(X=L)} + \\
&\quad + \log \frac{P(s_1, \ldots, s_m | X=H)}{P(s_1, \ldots, s_m | X=L)} \\
&= \theta(s_1, \ldots, s_m) \\
\end{align*}
\]

Type of \( \{s_1, \ldots, s_m\} \).

Law of large numbers

\[
\begin{align*}
\Rightarrow \theta(s_1, \ldots, s_m) &= \sum_{k=1}^{m} \theta(s_k) \\
&\xrightarrow{m \to \infty} \begin{cases} 
+\infty, & X = H \\
-\infty, & X = L
\end{cases}
\end{align*}
\]
Dynamics of one agent:

- Initially, agent $i$ given distinct subset $S_i \subset S = \{s_1, s_2, s_3, \ldots \}$.

- Agents in market randomly matched according to Poisson process, w/ common rate $\lambda$ across agents.

- Upon interacting, agents share signal sets:

  \[ \Theta_i(t) = \Theta_i(t-) + \Theta_j(t-) \]
  \[ \Theta_j(t) = \Theta_i(t-) + \Theta_j(t-) \]

  where

  \[ \Theta_i(t) = \Theta(S_i(t)) \], type of agent $i$ at $t \geq 0$. 
Information acquisition is costly — agents leave market when sufficiently informed.

Cost for individual agent:

\[
C_i(t) = \mathbb{E} \left[ \int_0^t e^{-rs} e^{-\mu s} ds + e^{-\mu t} \left( p_i(t) - 1 \{X = H\} \right)^2 \right]
\]

- Stopping time
- Discounting with rate \( \mu > 0 \)
- Running cost
- Quadratic exit cost (not known at exit!)

Where \( p_i(t) = P(X = H | S_i(t)) \in [0,1] \).

Can be written as

\[
C_i(t) = \mathbb{E} \left[ \int_0^t e^{-rs} e^{-\mu s} ds + e^{-\mu t} g(\theta_i(t)) \right]
\]

with \( g(\theta) = \frac{e^\theta}{(1 + e^\theta)^2}, \quad \theta \in (-\infty, \infty) \).

Choose \( t \) to minimize \( C_i \) → optimal stopping.
• Note: Cost function is given by expectation under Total Probability (i.e., not conditional on X).

\[
\mathbb{E} \left[ (\phi_i(t) - I_{\{X=H\}})^2 \right]
\]

\[
= \mathbb{E} \left[ (\phi_i(t) - 0)^2 (1 - \phi_i(t)) + (\phi_i(t) - 1)^2 \phi_i(t) \right]
\]

\[
= \mathbb{E} \left[ \phi_i(t) (1 - \phi_i(t)) \right]
\]

\[
= \phi(\theta_i(t))
\]

\[\Rightarrow \text{Cost penalizes "fence-sitting," not wrong answer given } X.\]
• **Mean-field:** \( \mu^X(t, d\theta) = \text{proportion of agents active in market at time } t > 0 \text{ with type in } d\theta, \) conditional on \( X \).

• \( \Theta_i(t) \) is a pure-jump (compound Poisson) Markov process w/ jump size distribution \( \mu^X(t, d\theta) \), given \( X \).

\[ L_{\mu^X(t)} \psi(\theta) = \lambda \int (\psi(\theta + \eta) - \psi(\theta)) \mu^X(t, d\eta). \]

\( \text{Test fcn.} \]

• Unconditional on \( X \), \( \Theta_i(t) \) has generator

\[ L_t = \frac{1}{2} L_{\mu^H(t)} + \frac{1}{2} L_{\mu^L(t)}. \]
Obstacle Problem Determines Stopping Region:

\[
\max \left\{ \partial_t v - \mathcal{L}_t v + \pi v - c, \quad v - g \right\} = 0
\]

Where we solve for Value Function \( v(t, \theta) \).

\( \mathcal{R}_t \) = Continuation Region = \( \text{supp} (v - g) \).

Stopping region at time \( t \geq 0 \) is \( \mathcal{R}_t^c \).
Forward Kolmogorov Eqn. Determines Evolution of Mean-Field:

If $P_i^x(t, d\Theta) = \text{dist. of } \Theta_i(t)$, conditionally on $X$

$$\partial_t P_i^x = \mathcal{L}_{\mu_i^x(t)} P_i^x \quad \text{w/ supp}(P_i^x) \subset \mathbb{R}_t$$

$\xrightarrow{\text{Lin}}$

$$\partial_t \mu^x = \mathcal{L}_{\mu^x} \mu^x \quad \text{w/ supp}(\mu^x) \subset \mathbb{R}_t$$

- This is a kinetic Eqn. on a bounded domain since

$$\partial_t \mu = \mathcal{L}_{\mu} \mu = \lambda \left[ \mu^* \mu - \mu \left( \int \mu \right) \right] \quad \text{on } \mathbb{R}_t.$$
To summarize:

\[
\begin{align*}
R_t &\rightarrow \{ \mu^H(t, \omega), \mu^L(t, \omega) \} \\
\text{FORWARD EQUATION (KINETIC)} &\rightarrow \{ \mu^H(t), \mu^L(t) \} \\
&\rightarrow \mathcal{R}_t^* \\
\text{AVERRAGING OVER VALUE OF } X \text{ (OBSSTACES)} &\rightarrow \mathcal{R}_t^*
\end{align*}
\]

\[\rightarrow \text{NASH/MFG EQUILIBRIA ARE FIXED POINTS OF THIS MAP!} \]
**Related Literature:**

*Previous work addresses optimal control of agents' matching rates.*

- **Duffie, Malamud, Manso** (*Econometrica, 2010*)
- **Aldous** (*arXiv, '10*), **Durrett, Chatterjee** (*Ann. Appl. Prob, '10*).

 ~> **Role of Network Geometry.**

- **Lasry, Lions, Guéant and co-authors** (*2006-*)

\[
\begin{align*}
-\nu \Delta m + H(x, \nabla u) + \lambda &= V(x, m) \quad \text{Backward Eqn. (HJB)} \\
-\nu \Delta m - \text{div} \left( \frac{\partial H}{\partial \phi} (x, \nabla u) m \right) &= 0 \quad \text{Forward Eqn. (Mean-Field)}
\end{align*}
\]

 ~> **Solutions are Nash/MFG Equilibria.**
STATIONARY PROBLEM:

- Agent $i$ replaced w/ new agent after time $T_i \sim \text{Exp} (\beta)$. New agent has initial type distributed according to input measures $\pi^X(d\theta)$, conditional on $X$.

- Assume $\pi^H(d\theta)$ symmetric to $\pi^L(d\theta)$.

**DEF. (NASH/MFG EQUILIBRIUM)**

$$(R, \mu^H) \text{ s.t. } \mu^H \geq 0, \supp(\mu^H) \subset R,$$

$$\max \left\{ -\mathbb{I} v + (\gamma + \beta) v - (c + \beta g), \ v - g \right\} = 0 \rightarrow \text{obstacle problem}$$

$$R = \supp (v - g).$$

$$0 = \chi (\mu^H \ast \mu^H - \mu^H (\int \mu^H)) + \beta (\pi^H - \mu^H) \text{ w } R \rightarrow \text{forward eqn.}$$
• Trivial Nash/MFG equilibrium \((R, \mu^H) = (\{\emptyset\}, 0)\).

• Nontrivial equilibria?

Depends on rates: \(\lambda, \beta, \mu\)

And costs: \(c, g(\theta)\)

And initial info.: \(\Pi^H\)

\(\Rightarrow\) Yes. (Indicated by numerics).
**Numerics:** Make ansatz \( R = (-r, r) \) for some \( r \geq 0 \).

- With \( \lambda = 2, \beta = 0.05, \gamma = 0.05, c = 0.0125 \),
  \( \Rightarrow 40 \) meetings/agent on ave.

\[
\pi^H(d\Theta) = \frac{\exp\left(-\frac{\Theta-1}{\beta}\right) d\Theta}{\int \exp\left(-\frac{\Theta-1}{\beta}\right) d\Theta},
\]

\( R \rightarrow R^* (r \rightarrow r^*) \).

**Largest equilibrium value is Pareto-optimal!**
Forward Scheme:

\[
\begin{align*}
0 &= \lambda (\mu \ast \mu - \mu (S^r \mu)) + \beta (\pi - \mu) \quad \text{in } (-r, r) \\
\text{supp } \mu &\subset (-r, r).
\end{align*}
\]

\Rightarrow \text{ Contractive map for } \mu \text{ when } \beta > 4 \lambda.

- For general \( \beta, \lambda > 0 \) write as \( \mu = \Phi(\mu), \)

\[
\Phi(\mu) = \frac{1}{\beta + \lambda (S^r \mu)} \left( \lambda (\mu \ast \mu) + \beta \pi \right) \mathbb{1}_{(-r, r)}
\]

- \[
\begin{align*}
\mu^{(n+1)} &= \Phi(\mu^{(n)}) \\
\mu^{(0)} &= \pi \mathbb{1}_{(-r, r)}
\end{align*}
\]

converges for any \( \beta, \lambda > 0 \)

(proof in progress)
Obstacle Problem Scheme:

- Random matching $\implies$ Pure-jump process (compound Poisson) for individual's type.

  $\implies$ Agents only stop immediately after jump (unless replaced before then).

$\implies$ Time-discrete problem!

Let

$$T \psi (\Theta) = \mathbb{E} \left[ \int_0^{\tau_1} e^{-(\delta + \beta)s} \left( c + \beta g(\Theta_i(s)) \right) ds + e^{-(\delta + \beta)\tau_1} \psi (\Theta_i(\tau_1)) \right]$$

where $\tau_1 \sim \text{Exp}(\lambda)$ is first jump time of $\Theta_i(t)$. 
\[
\max \left\{ -Lv + (H + \beta) v - (c + \beta g), \ v - g \right\} = 0
\]

\[\iff\]

\[
\min \left\{ TV, g \right\} = v
\]

**WALD-BELLMAN EQN.**

**SCHEME:**

\[
\begin{aligned}
\nu^{(n+1)} &= \min \left\{ TV^{(n)}, g \right\} \\
v^{(0)} &= g
\end{aligned}
\]

\[\Rightarrow \nu^{(n)} \downarrow v \ \text{UNIF.}\]

**SAME PROPERTY THAT LEADS TO KINETIC FORWARD EQN.**

**SIMPLIFIES OPTIMAL STOPPING PROBLEM.**
Model allows for rich class of equilibria

Ex. (Educated get richer, poorly educated get poorer)

$Q(\theta)$

$\Pi^H(\theta)$

Badly educated remain and leave with wrong information.

Educated leave market immediately.
Work in Progress:

- Existence Theorem for Nash/MFG Equilibria.

- How "bad" can $\Omega$ be? 
  \[ \rightarrow \text{as "bad" as obstacle } \phi \]

- For $g$ similar to one originally considered 
  \[ g'' \text{ has only one change in sign for } \theta > 0 \]
  Hypothesize that 
  \[ R = (-\bar{r}, \bar{r}) \setminus (-\bar{r}_\text{inner}, \bar{r}_\text{inner}) \]
  for some $\bar{r}, \bar{r}_\text{inner} > 0$. 

\[ R \]
Intuition:

- Those agents close to sudden drop in exit cost stay active in market.
- Those completely uninformed give up immediately.
Thanks for listening!